Angular transport in a nonperiodic Chirikov-Taylor map

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Transport in angular direction is considered for a nonperiodic Chirikov-Taylor (standard) map. In the limit of large stochasticity parameter, depending on the boundary conditions of the action variable, either superdiffusive of diffusive behavior is found. In both cases characteristic oscillations in the transport coefficients occur. Theoretical predictions based on the Perron-Frobenius evolution operator formalism for the distribution function are compared with numerical simulations. Information on the anomalous behaviors in the near threshold as well as in the subthreshold regions are also presented.

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I. INTRODUCTION

The Chirikov-Taylor map [1], also called standard map, is a single parameter nonlinear twist map that describes the local behavior of nonintegrable dynamical systems in the separatrix region of nonlinear resonances [1–5]. It has become a paradigm for investigating the properties of chaotic dynamics in Hamiltonian systems.

In the past many authors investigated the transport in chaotic systems on the basis of the standard map. In the limit of large stochasticity parameter K, the standard map exhibits a *diffusive behavior* in the action variable p, with a quasioscillating diffusion constant D_p as a function of K. The latter was first numerically discovered by Chirikov [1]. Rechester and White [6] and Rechester *et al.* [7] used a probalistic method for the solution of the Vlasov equation and showed that the behavior of test particles becomes diffusive (in p) for very large stochasticity parameters. Their calculation of a turbulent diffusion coefficient was refined by, e.g., Hasegawa and Saphir [8,9]. Balescu and co-workers [10,11] used a continuous time random walk model for subcritical values of the stochasticity parameter, finding subdiffusive motion in that parameter region. Benkadda *et al.* [12] found superdiffusive transport caused by accelerator islands. The anomalous exponents were related to the characteristic temporal and spatial scaling parameters of the island chain. White *et al.* [13] investigated anomalous transport near threshold. Very long flights and a large anomaly in the transport were shown to be associated with multiisland structures causing orbit sticking. Khodas and Fishman [14] and Khodas *et al.* [15] used the Frobenius-Perron operator formalism for the kicked rotor, which can be considered as the continuous analog of the discrete standard map, in order to calculate relaxation and diffusion in the presence of noise. Bénisti and Escande [16] showed that the diffusion properties of the standard map are nonuniversal in the framework of the wave-particle interaction.

The standard map may be written in action-angle variables p and θ,

$$p' = p - \frac{K}{2\pi}\sin 2\pi\theta, \tag{1}$$

$$\theta' = \theta + p', \tag{2}$$

where K is the stochasticity (control) parameter and

$$0 \leqslant \theta < 1, \quad -\infty < p < \infty. \tag{3}$$

Because of the obvious translation symmetry it is convenient to study the topological properties of the phase plane with boundary conditions on the torus, e.g.,

$$0 \leqslant \theta < 1, \quad -0.5 \leqslant p < 0.5. \tag{4}$$

The topologies of the Poincaré plots for sets of numerically iterated trajectories depend on the stochasticity parameter K. A threshold value K_c exists. For K values around K_c characteristic peculiarities such as stochastic sea, island chains, KAM surfaces, etc., can be clearly seen.

Another choice of boundary conditions (4) originates from the specific applications one has in mind. For example, in plasma physics, the model was extensively used in order to understand generic behavior of field line and orbit dynamics in partially chaotic as well as completely chaotic magnetic systems. Then, for tokamak applications, in the two-dimensional dynamical system p can be considered as the ''radial'' coordinate whereas θ denotes the poloidal angle coordinate (measured in radians divided by 2π). All values are then taken at discrete ''times'' ν, which correspond to values of the toroidal angle. The latter is assumed as nonperiodic while the poloidal angle is periodic, as denoted above [6,7,11]. We should note that in the area of stochastic magnetic field line transport, later some more sophisticated models than the standard map have been used [10,17–19].

However, other boundary conditions are also in use. When considering the kicked rotator on the torus [14,15], one takes θ as 1 periodic and p as s periodic, where s is an integer.

In numerous publications on transport in the standard map, the considerations concern only the transport in p direction. However, this map being asymmetric, it is of interest to consider both directions, p and θ, in transport computations [12]. In some physical applications one needs the angle evolution.

In this paper we investigate the cases of infinite phase space for both θ and p in more detail. To be more specific, we apply two types of boundary conditions,

$$\text{case A:} \quad -\infty<\theta<\infty, \quad -\infty<p<\infty, \tag{5}$$

$$\text{case B:} \quad -\infty<\theta<\infty, \quad -0.5\leqslant p<0.5; \tag{6}$$

thus in case B we use p modulo 1.

In the following we shall use Eqs. (1) and (2) together with the just-mentioned restrictions A or B, respectively.

For large K and in a first approximation, the action diffusion coefficient D_p of Eqs. (1)–(3) can be estimated as

$$D_p \approx \frac{K^2}{4}[1-2J_2(K)], \tag{7}$$

where J_2 is the Bessel function of the first kind (and second order). However, for some values of K the system exhibits a much more complicated dynamics. The presence of ‘‘survived’’ islands, surrounding stable periodical points, makes the phase space, in contrast to ‘‘pure stochastic’’ systems, a mixture of regular and chaotic components. The stickiness property leads to strong deviations from the diffusive law. This phenomenon is called anomalous transport. In its long time asymptotics $\nu\to\infty$, the stochastic system can be characterized by the transport exponents μ_p (for motion in p direction) and μ_θ (for motion in θ direction) that are defined through

$$\langle(\Delta p)^2\rangle \sim \nu^{\mu_p}, \quad \langle(\Delta\theta)^2\rangle \sim \nu^{\mu_\theta}, \tag{8}$$

respectively. Following the usual notation, the transport regime can be characterized as

$$\mu=1, \quad \text{diffusive,}$$

$$\mu<1, \quad \text{subdiffusive,}$$

$$\mu>1, \quad \text{superdiffusive.}$$

It was shown that at some values of K the islands have strong effects on transport. For example, the two cases, K near the threshold $K_c\approx 0.971$ and K near $K_n=2\pi n$, $n=1,2,\ldots$, respectively, were considered. In the latter case, relatively large accelerator mode islands lead to ‘‘peaks’’ of the transport exponent.

The object of our interest is the transport in the *angular* direction for both cases A and B, i.e., to find, if they exist, corresponding expressions for D_θ. The present investigation deals with the asymptotic behaviors of the system as $K\to\infty$ and $K\to 0$, respectively. Divergences, arising from accelerator mode stable points and other types of stable periodical points will also be discussed.

Let us briefly mention some expected differences between transport in p and θ directions, respectively.

First a summary of the p transport. For $K<K_c$, transport barriers in p direction exist in the form of KAM surfaces. Just above the threshold K_c, the system exhibits a very complicated phase space topology. The multiisland structure of the phase plane causes orbit sticking, which leads to changes in transport rate. The cross section of a capture to the sticky part of the phase space has a very sensitive dependence on K. By investigations similar to those of White *et al.* [13] one might show a strange (subdiffusive) behavior with strongly varying exponents near threshold. For $K\gg K_c$, as the structures by islands vanish, the transport in p becomes diffusive. The leading order of expression (7) immediately follows from

$$\langle(\Delta p)^2\rangle \sim \frac{(K/2\pi)^2}{4}2\nu \sim \frac{1}{2\pi^2}D_p\nu, \tag{9}$$

where $\Delta p=p_n-p_0$. Here, $\langle\cdots\rangle$ means averaging over different initial conditions.

Now the different expectations for the (mostly unknown) situation of transport in θ direction. Even for $K<K_c$ we expect a (strong) superdiffusive transport. This expectation is motivated by the following thought experiment. Let $K=0$ and initial conditions being distributed on p axis as $n(p)\delta(\theta)$. After ν iterations the latter becomes $n(p)\delta(\theta-\nu p)$. Thus, the mean square displacement shows the following behavior:

$$\langle(\Delta\theta)^2\rangle = \int\int \theta^2 n(p)\delta(\theta-\nu p)dp\,d\theta$$
$$= \int (\nu p)^2 n(p)dp \sim \nu^2. \tag{10}$$

Near the threshold K_c we also expect a strange behavior. Extrapolating from the derived variations of the exponents for case B [13], we expect that for the latter case and $K\gg K_c$ the transport may be diffusive. In case A, on the other hand, the transport may be much faster, e.g., $D_\theta\sim D_p\nu^2$ (see below).

The paper is organized as follows. To analyze the angular transport, here we apply a procedure based on the Frobenius-Perron operator. In Sec. II, we briefly outline the method. Section III is devoted to the behavior in the limit $K\to 0$. The opposite limit $K\to\infty$ is considered in two sections. First, we treat case A in Sec. IV. The case B is evaluated in Sec. V. The paper is concluded by a short summary and discussion.

II. OUTLINE OF THE METHOD

In this section we summarize the definitions and formulas of the Frobenius-Perron operator formalism that are necessary for an understanding of the main results of the present paper. For reasons of simplicity, we choose here case A for demonstration.

Let us characterize the state of the system by a distribution function $f(p,\theta)$. Provided the latter is known at ‘‘time’’ ν, the state of the system at ‘‘time’’ $\nu+1$ then follows from the equations of motion. Contrary to ‘‘ordinary’’ statistical mechanics, in the dynamical systems based on maps one introduces discrete ‘‘times’’ ν. The evolution in ‘‘time’’ of the distribution function will be represented via the Frobenius-Perron operator [9].

For the distribution function, an ensemble of orbits, each governed by the standard map, is considered. The thereby defined (coarse-grained) distribution function should be smooth and integrable in the limit of an infinite ensemble. Although in the case of weak chaos, KAM surfaces prevent orbits from reaching arbitrary values of p, it is assumed that $f=f(p,\theta;\nu)$ is defined in the whole phase plane (θ,p).

The local density $n(\theta;\nu)$ is introduced as the average of $f(p,\theta;\nu)$ over the action variable p,

$$n(\theta;\nu)=\int_{-\infty}^{\infty}dp\, f(p,\theta;\nu). \tag{11}$$

Next, we switch to the Fourier-transform in θ and p; we use the definition

$$\tilde{f}(q,m;\nu)=\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}d\theta \exp[-2\pi i(pq+\theta m)]f(p,\theta;\nu). \tag{12}$$

Here, q and m as well as q' and m' are continuous variables since we define p and θ on the whole plane.

The Fourier transformation of the density is related to the $q=0$ mode of the Fourier-transformation of $f(p,\theta;\nu)$,

$$\tilde{n}(m;\nu)=\tilde{f}(q=0,m;\nu). \tag{13}$$

In the statistical theory of transport, the diffusion coefficient is related to the time-dependent mean-square-displacement (MSD). The latter we define via

$$\Sigma_{\theta}^{2}(\nu)=\int_{-\infty}^{\infty}d\theta\,\theta^{2}n(\theta;\nu); \tag{14}$$

Σ_{θ}^{2} tends to $\langle(\Delta\theta)^{2}\rangle$ for $\nu\rightarrow\infty$.

A simple calculation leads to

$$\left.\frac{d^{2}\tilde{n}(m;\nu)}{dm^{2}}\right|_{m=0}=-4\pi^{2}\Sigma_{\theta}^{2}(\nu). \tag{15}$$

Thus, only an infinitely small region of $\tilde{n}(m;\nu)$ near $m=0$ determines the MSD [provided $\tilde{n}(m;\nu)$ is an analytic function]. Obviously, this means that for long times only the lowest m modes of the distribution function determine the MSD.

It is natural to define the *running* diffusion coefficient $D_{\theta}=D_{\theta}(\nu)$ as a time derivative of the displacement $\Sigma_{\theta}^{2}(\nu)$, i.e.,

$$D_{\theta}=2\pi^{2}\frac{\partial\Sigma_{\theta}^{2}}{\partial\nu}=-\frac{1}{2}\frac{\partial}{\partial\nu}\left.\frac{d^{2}\tilde{n}(m;\nu)}{dm^{2}}\right|_{m=0}. \tag{16}$$

The evolution of the distribution function can be calculated by the action of an evolution operator $\hat{U}$, called the Frobenius-Perron operator,

$$f(p',\theta';\nu)=f(p,\theta,\nu-1)=\hat{U}f(p',\theta';\nu-1). \tag{17}$$

Because of the explicit inversion formulas

$$p=p'+\frac{K}{2\pi}\sin 2\pi(\theta'-p'), \tag{18}$$

$$\theta=\theta'-p', \tag{19}$$

for the standard map, the Frobenius-Perron operator can be expressed explicitly as the simple finite displacement operator

$$\hat{U}=\exp\left(-p\frac{\partial}{\partial\theta}\right)\exp\left(\frac{K}{2\pi}\sin 2\pi\theta\frac{\partial}{\partial p}\right). \tag{20}$$

In the Fourier representation, the operator $\hat{U}$ has the matrix elements

$$\langle q,m|\hat{U}|q',m'\rangle=\int_{-\infty}^{\infty}dp\int_{-\infty}^{\infty}d\theta \exp[-2\pi i(pq+\theta m)]\times\hat{U}\exp[2\pi i(pq'+\theta m')]. \tag{21}$$

Using these matrix elements, the evolution of the distribution function in Fourier space becomes

$$\tilde{f}(q,m;\nu)=\int_{-\infty}^{\infty}dq'\int_{-\infty}^{\infty}dm'\langle q,m|\hat{U}|q',m'\rangle\times\tilde{f}(q',m';\nu-1). \tag{22}$$

It is an important fact of the standard map that the matrix elements of the Frobenius-Perron operator can be calculated analytically. Using the well-known identity with Bessel functions J_l (first kind, lth order)

$$\exp(iz\sin\varphi)=\sum_{l=-\infty}^{\infty}e^{il\varphi}J_{l}(z), \tag{23}$$

one obtains

$$\langle q,m|\hat{U}|q',m'\rangle=J_{m-m'}(q'K)\,\delta(q+m-q')\times\prod_{l=0,\pm1,\ldots}\delta(m-m'-l). \tag{24}$$

Although m and m' are continuous, the difference $m-m'$ remains integer. This reflects the fact, that the Frobenius-Perron operator is invariant under translation transformations $\tilde{\theta}=\theta+n, n\in Z$.

Making use of the explicit form (24), integration over q' and m' can be replaced by summation in following way:

$$\int\int dq'dm'\,\delta(q+m-q')\prod_{l=0,\pm1,\ldots}\delta(m-m'-l)\cdots\rightarrow\sum_{l}\delta_{q',q+m}\delta_{m',m-l}\cdots.$$

Thus, we can write

$$\tilde{f}(q,m;\nu)=\sum_{l=0,\pm 1,\ldots} J_l(q'K)\tilde{f}(q'=q+m,m'=m-l;\nu-1). \tag{25}$$

The solution of the initial value problem in Fourier space follows as

$$\tilde{f}(q,m;\nu)=\int dq'\int dm'\langle q,m|\hat{U}^{\nu}|q',m'\rangle\tilde{f}(q',m';0). \tag{26}$$

In a similar manner as above, one can write the Fourier-transformation of the density as

$$\tilde{n}(m;\nu)=\int dq'\int dm'\langle q=0,m|\hat{U}^{\nu}|q',m'\rangle\tilde{f}(q',m';0). \tag{27}$$

One can show

$$\langle 0,m|\hat{U}^{\nu}|q',m'\rangle =\int dq_1\cdots\int dm_{\nu-1} \times\langle 0,m|\hat{U}|q_1,m_1\rangle\cdots\langle q_{\nu-1},m_{\nu-1}|\hat{U}|q',m'\rangle. \tag{28}$$

Introducing new summation indices $k_i=m_i-m, i=1,\ldots,\nu-1$ and keeping in mind an integration over q' and m', the propagator finally becomes

$$\langle q,m|\hat{P}\hat{U}^{\nu}|q',m'\rangle =\sum_{k_1,k_2,\ldots,k_{\nu-1}} J_{k_0-k_1}(mK)J_{k_1-k_2}[(2m+k_1)K] \times\cdots J_{k_{\nu-1}-k_\nu}[(\nu m+k_1+k_2+\cdots+k_{\nu-1})K] \times\delta_{m',k_\nu-m}\delta_{q',\nu m+k_1+k_2+\cdots+k_{\nu-1}}, \tag{29}$$

where the $k_i, i=0,\ldots,\nu$, are integers with $k_0=0$, $k_\nu=m'-m$.

If we do not specify q' and m', i.e., the matrix element, then each string $k_1,\ldots,k_\nu$ determines to which matrix element the corresponding term in the sum (29) contributes. On the other hand, for a given matrix element the two conditions $q'=\nu m+k_1+\cdots+k_{\nu-1}$ and $k_\nu=m'-m$ must hold.

Equation (29) is an exact result that provides the starting point for an asymptotic analysis.

III. ASYMPTOTIC ANALYSIS FOR $K\to 0$

We shall show now how for $K<K_c$ the transport behavior in θ direction can be obtained from the propagator (29) in the limit $K\to 0$. In Appendix A we present another (simpler) derivation based on the solution of the continuous system of a kicked rotor that corresponds to the discrete standard map.

The propagator (29) consists of a product of Bessel functions. All the arguments q_iK are small in the limit $K\to 0$. We, therefore, expand the Bessel functions in the form

$$J_n(x)=x^n\left[\frac{1}{2^n\Gamma(n+1)}-\frac{x^2}{2^{n+2}\Gamma(n+2)}+O(x^4)\right]. \tag{30}$$

It is easy to see that the main contribution in the sum (29) originates from the term with

$$k_1=k_2=\cdots=k_{\nu-1}=k_\nu=0. \tag{31}$$

In the case $K=0$ this is the only nonvanishing term. It is of zeroth order in K; terms with any other choice of k_i contain higher orders of K.

Thus, in zeroth order we have

$$\langle q,m|\hat{P}\hat{U}^{\nu}|q',m'\rangle=1 \tag{32}$$

for $q'=\nu m$, and $m'=m$. The solution of the equation of motion (27) becomes

$$\tilde{n}(m;\nu)=\tilde{f}(\nu m,m;0). \tag{33}$$

This is exactly what we get in a corresponding continuous-time system for $K=0$ (see Appendix A). Thus, the regime at $K\to 0$ is superdiffusive with respect to angular transport, and the diffusion coefficient is

$$D_\theta=\text{const}\quad\nu, \tag{34}$$

where const depends on the initial distribution [see Eq. (A9)]. This is obvious from the physical point of view. At $K=0$ the ''orbits'' do not change their momenta p at all. So the rate of transport depends only on how the orbits are initially distributed in p.

It should be emphasized that the predicted value $\mu_\theta=2$ (for $K\ll K_c$) holds for both cases A and B.

IV. ASYMPTOTIC ANALYSIS IN THE LIMIT $K\to\infty$ FOR CASE A

Here, our aim is to find a diffusion coefficient in the limit $K\to\infty$. Given the solution $\tilde{n}(m;\nu)$, we have to differentiate it twice at $m=0$. As was already mentioned, the behavior of $\tilde{n}(m;\nu)$ only in a small region near 0 is of interest. Although we consider large K, for any fixed K the limit $m\to 0$ can be applied. Thus, in the following two assumptions are crucial. First, we assume a large stochasticity parameter, i.e.,

$$K\gg 1. \tag{35}$$

But second, we can assume sufficiently small m, i.e.,

$$mK\ll 1. \tag{36}$$

A. Lowest order in $1/\sqrt{K}$

There are two types of arguments in the Bessel functions of the propagator (29): small ones, if they are like imK, and large ones, if they are like $(im+j)K$, where i,j are integers.

Apart from that, we note that for diagonal elements only the terms being proportional to m^2 give a nontrivial contribution to the MSD [see Eq. (15)].

For large arguments we use the expansion

$$J_n(x)=\sqrt{\frac{2}{\pi x}}\left[\cos\left(x-\frac{\pi n}{2}-\frac{\pi}{4}\right)+O\left(\frac{1}{x}\right)\right]. \quad (37)$$

We shall expand the propagator in a power series of two small parameters, $\sqrt{1/K}$ and mK, respectively.

Since we consider the case A (which has been outlined in Sec. II) we can immediately proceed with the expressions derived in Sec. II. Again, the dominant term in the sum (29) is that one corresponding to

$$k_1=k_2=\cdots=k_{\nu-1}=k_\nu=0. \quad (38)$$

We shall designate it by S_0. Expanding the Bessel functions up to the second order in mK, we find

$$S_0=J_0(mK)J_0(2mK)\cdots J_0(\nu mK)$$
$$\approx\left(1-1\frac{m^2K^2}{4}\right)\left(1-2^2\frac{m^2K^2}{4}\right)\cdots\left(1-\nu^2\frac{m^2K^2}{4}\right). \quad (39)$$

Up to the second order this product leads to

$$S_0\approx 1-\frac{m^2K^2}{4}\sum_{i=1}^{\nu} i^2=1-\left(\frac{\nu}{6}+\frac{\nu^2}{2}+\frac{\nu^3}{3}\right)\frac{m^2K^2}{4}. \quad (40)$$

In the next step we consider the transition to large ''times'' (the limit $\nu\to\infty$ should be taken for fixed K),

$$S_0\approx-\frac{\nu^3}{3}\frac{m^2K^2}{4}. \quad (41)$$

This contribution belongs to the matrix element $\langle q=0,m|\hat{U}^\nu|q',m'\rangle$ with $q'=\nu m$; $m'=m$. The next terms we consider will contain the additional small factor $J_2(K)\sim\sqrt{1/K}$. We designate these terms as

$$S_i: \quad k_1=\cdots=k_{i-1}=0,\ k_i=1,\ k_{i+1}=-1,$$
$$k_{i+2}=\cdots=k_{\nu-1}=0;$$
$$S_{-i}: \quad k_1=\cdots=k_{i-1}=0,\ k_i=-1,\ k_{i+1}=1,$$
$$k_{i+2}=\cdots=k_{\nu-1}=0;$$

for $i=1,2,\ldots,\nu-2$.

In dominant order, S_i can be estimated as

$$S_i=i(i+2)\frac{m^2K^2}{4}J_2(K), \quad (42)$$

and we have $S_{-i}=S_i$. Since all these terms contribute to the same matrix element, we can add them together, with the result

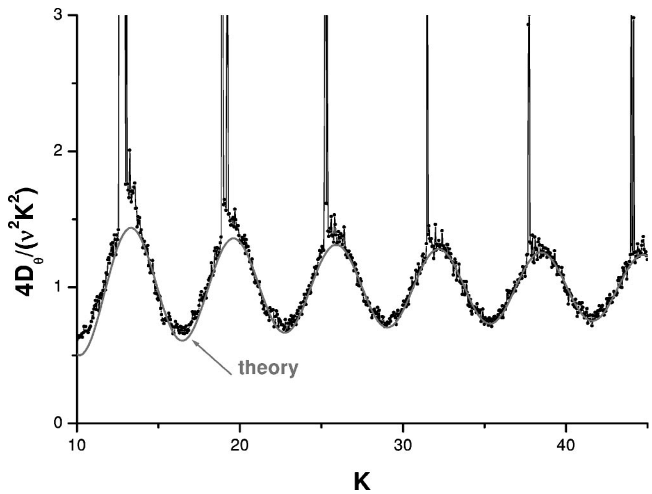

FIG. 1. Normalized diffusion coefficient D_θ vs control parameter K for case A $(-\infty<p<+\infty)$. The dots (connected by lines) represent the measurements from numerical simulations while the solid curve shows the theoretical prediction.

$$\sum_{i=1}^{\nu-2}(S_i+S_{-i})=2\frac{\nu^3}{3}\frac{m^2K^2}{4}J_2(K). \quad (43)$$

Here the summation formula

$$\sum_{i=1}^{\nu-2} i(i+2)=1-\frac{5\nu}{6}-\frac{\nu^2}{2}+\frac{\nu^3}{3}\to\frac{\nu^3}{3}, \quad \text{for} \quad \nu\to\infty, \quad (44)$$

was used. Combining with S_0, we finally get for the matrix element

$$\langle q,m|\hat{P}\hat{U}^\nu|\nu m,m\rangle=-\frac{\nu^3}{3}(1-2J_2(K))\frac{m^2K^2}{4}. \quad (45)$$

We note that for θ transport asymptotically the propagator has off-diagonal elements, and it *cannot* be represented in the form

$$\langle q,m|\hat{P}\hat{U}^\nu|q',m'\rangle\sim\delta(q'-q)\,\delta(m'-m), \quad (46)$$

as it was in the case for ''action'' diffusion.

The propagator (45) delivers the explicit solution of the equation of motion (27),

$$\tilde{n}(m;\nu)=-\frac{\nu^3}{3}(1-2J_2(K))\frac{m^2K^2}{4}\tilde{f}(\nu m,m;0). \quad (47)$$

Now, combining Eqs. (15) and (16), the running diffusion coefficient can be estimated as

$$D_\theta\approx\frac{K^2}{4}(1-2J_2(K))\nu^2\approx\frac{K^2}{4}\left(1+\sqrt{\frac{8}{\pi K}}\cos\left[K-\frac{\pi}{4}\right]\right)\nu^2. \quad (48)$$

In case A the θ transport is superdiffusive and the running diffusion coefficient is proportional to ν^2.

Figure 1 shows a comparison of the analytical prediction with numerical simulations. Clearly, the oscillations with K can be seen in D_θ/ν^2. The diffusion coefficient in angle variable is related to that in action variable (7) via

$$D_\theta = D_p \nu^2. \tag{49}$$

In Appendix B we present simple arguments to explain why that should be the case.

B. Influence of accelerator mode islands

The present analysis does not apply to the regions with periodical points. We mention in that respect the so-called accelerator modes, which play a special role in diffusion problems. Divergences in the $D(K)$ dependence can be observed. They are related to ''accelerated'' islands, consisting of elliptic orbits surrounding an accelerator mode fixed point [3].

For a period-1 accelerator mode point (p_0,θ_0) [period 1 means that in a ''moduled'' map it would be a period-1 fixed point] we have the condition

$$-\frac{K}{2\pi}\sin 2\pi\theta_0 = N, \quad N\in \mathbf{Z}; \tag{50}$$

the starting momentum p_0 has to be integer, say 0. The action p, starting in this point, increases at each step by N, and θ grow as ν^2,

$$p_\nu = p_0 + \nu N = \nu N, \tag{51}$$

$$\theta_\nu = \theta_0 + p_1 + \cdots + p_\nu \rightarrow \frac{\nu^2}{2}N. \tag{52}$$

Therefore, the mean square displacement in θ direction increases as ν^4. For the transport exponents one gets in these accelerated regions of phase space $\mu_p=2$, $\mu_\theta=4$. The relation $\mu_\theta=\mu_p+2$, discussed in Appendix B, is therefore also valid here. Even when only a few orbits are located in the accelerator mode island, their contribution to the MSD becomes dominant as $\nu\rightarrow\infty$.

A linear stability analysis of the accelerator mode islands shows the stability windows

$$\frac{1}{4}-\frac{1}{\pi^2 N} < \theta_0 < \frac{1}{4}, \tag{53}$$

$$K=\frac{2\pi N}{\sin 2\pi\theta_0}. \tag{54}$$

The reason why the accelerator mode contributions were not evident in the propagator expansion is the following. The Fourier modes have been calculated to the lowest order with respect to the small parameter $\sqrt{1/K}$. Contributions of accelerator modes are of higher order in $\sqrt{1/K}$, but contain an additional factor ν. That is why they become dominant as $\nu\rightarrow\infty$.

Although the stability windows for the accelerator islands are very narrow, there exist stable accelerator mode islands of higher periods. Their existence would lead to the conclusion that divergence should take place for most (probably for all) values of K. In transport simulations that is not observed. As far as we deal with a finite number of orbits, there is a finite probability that an arbitrary orbit can be found in one of the accelerator mode islands. The probability is proportional to the areas occupied by the latter. In turn, these areas (similar to areas of any structures in phase plane) tend to zero as the stochasticity parameter increases. In order to observe the corresponding divergences in computer simulations, either a huge number of orbits with random initial conditions must be taken, or one has to choose some (quite specific) initial conditions on the islands.

To avoid the divergence difficulty, a collision term could be added to the Liouville equation, describing a phase flow of a continuous-time system [6,15] (see Appendix A):

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\{H,f\}=\frac{a^2}{2}\frac{\partial^2 f}{\partial\theta^2}, \tag{55}$$

where a^2 characterizes the diffusion process. The latter leads to an additional factor $\exp(-(a^2/2)m^2)$ in the propagator (24). As a consequence, the propagator does not lead anymore to matrix elements with rapid time growth rates caused by accelerator mode (quasi-)periodical orbits. After performing the calculations with diffusion, the limit of vanishing noise $(a\rightarrow 0)$ can be taken.

V. ASYMPTOTIC ANALYSIS IN THE LIMIT $K\rightarrow\infty$ FOR CASE B

We now consider the dynamics with periodicity in p, i.e., $-0.5\leqslant p<0.5$, $-\infty<\theta<\infty$. In order to apply the Frobenius-Perron operator formalism, which essentially needs the explicit inversion of the map, let us introduce

$$[p]:=p \bmod 1, \tag{56}$$

where the operation ''mod'' forces p to the interval $[-0.5,0.5[$. Now we can introduce the new map

$$p' = p-\frac{K}{2\pi}\sin 2\pi\theta, \tag{57}$$

$$\theta' = \theta + [p'], \tag{58}$$

which has the same angular dynamics as Eqs. (1) and (2) have for $-0.5\leqslant p<0.5$, $-\infty<\theta<\infty$. Modifying the previous calculations, we get for the Fourier representation of Frobenius-Perron operator

$$\langle q,m|\hat{U}|q',m'\rangle = J_{m-m'}(q'K)\int_{-\infty}^{\infty}dp \times\exp(2\pi i\{(q'-q-m+m')p-m'[p]\}) \times\prod_{l=0,\pm1,\ldots}\delta(m-m'-l). \tag{59}$$

The function $[p]$ can be written as a Fourier series,

$$[p]=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\pi k}\sin 2\pi kp. \tag{60}$$

Using this and the summation formula (23), the one-step propagator $\langle q,m|\hat{U}|q',m'\rangle$ becomes

$$\langle q,m|\hat{U}|q',m'\rangle=\sum_{k_1,k_2,\ldots}J_{m-m'}(q'K)J_{k_1}\left(\frac{2m'}{1}\right)\times J_{k_2}\left(\frac{2m'}{2}\right)\cdots\prod_{l=0,\pm1,\ldots}\delta(m-m'-l)\times\delta(q'-q-m+m'-k_1+2k_2-3k_3+\cdots), \tag{61}$$

where the summation indices k_i are integers. Next, we multiply the propagator ν times to obtain the ν-step propagator

$$\langle q,m|\hat{P}\hat{U}^{\nu}|q',m'\rangle=\int dq_1\cdots\int dm_{\nu-1}\times\langle 0,m|\hat{U}|q_1,m_1\rangle\cdots\langle q_{\nu-1},m_{\nu-1}|\hat{U}|q',m'\rangle. \tag{62}$$

Again, we have convolutions over all intermediate indices $q_1,m_1,\ldots,q_{\nu-1},m_{\nu-1}$. To be more specific, introducing $l^1=m-m_1, l^2=m_1-m_2,\ldots,$ and using upper indices $1,2,\ldots,\nu$ to distinguish the contributions from the various factors, one finds explicitly

$$\langle q,m|\hat{P}\hat{U}^{\nu}|q',m'\rangle=\sum_{l^1,k_1^1,k_2^1,\ldots}\sum_{l^2,k_1^2,k_2^2,\ldots}\cdots\sum_{l^{\nu},k_1^{\nu},k_2^{\nu},\ldots}J_{l^1}(Kq_1)J_{k_1^1}\left(\frac{2m_1}{1}\right)J_{k_2^1}\left(\frac{2m_1}{2}\right)J_{k_3^1}\left(\frac{2m_1}{3}\right)\cdots$$
$$\times J_{l^2}(Kq_2)J_{k_1^2}\left(\frac{2m_2}{1}\right)J_{k_2^2}\left(\frac{2m_2}{2}\right)J_{k_3^2}\left(\frac{2m_2}{3}\right)\cdots$$
$$\vdots$$
$$\times J_{l^{\nu}}(Kq_{\nu})J_{k_1^{\nu}}\left(\frac{2m_{\nu}}{1}\right)J_{k_2^{\nu}}\left(\frac{2m_{\nu}}{2}\right)J_{k_3^{\nu}}\left(\frac{2m_{\nu}}{3}\right)\cdots\delta_{q',q_{\nu}}\delta_{m',m_{\nu}}, \tag{63}$$

where

$$q_n=\sum_{i=1}^{n}(l^i+k_1^i-2k_2^i+3k_3^i-\cdots),$$

$$m_n=m-\sum_{i=1}^{n}l^i,$$

$$n=1,\ldots,\nu. \tag{64}$$

We consider now the limit $K\to\infty$. We start with the zeroth order terms (in $\sqrt{1/K}$) in a power series expansion of the propagator. The lowest order follows from the ‘‘trivial’’ choice of all coefficients

$$l^i=k_j^i=0;\quad i=1,2,\ldots,\nu;\quad j=1,2,\ldots. \tag{65}$$

Then we get $q_n=0$, $m_n=m$, for all n (the dominant matrix element is that with $q'=0$, $m'=m$), and

$$\langle q,m|\hat{P}\hat{U}^{\nu}|q',m'\rangle\approx\langle 0,m|\hat{U}^{\nu}|0,m\rangle\approx J_0\left(\frac{2m}{1}\right)J_0\left(\frac{2m}{2}\right)J_0\left(\frac{2m}{3}\right)\cdots\times J_0\left(\frac{2m}{1}\right)J_0\left(\frac{2m}{2}\right)J_0\left(\frac{2m}{3}\right)\cdots$$
$$\vdots$$
$$\times J_0\left(\frac{2m}{1}\right)J_0\left(\frac{2m}{2}\right)J_0\left(\frac{2m}{3}\right)\cdots. \tag{66}$$

Expanding the Bessel functions,

$$J_0(m)\approx 1-\frac{m^2}{4}, \tag{67}$$

we get for each row

$$J_0\left(\frac{2m}{1}\right)J_0\left(\frac{2m}{2}\right)J_0\left(\frac{2m}{3}\right)\cdots\approx 1-m^2\sum_{i=1}^{\infty}\frac{1}{i^2}=1-\frac{\pi^2}{6}m^2. \tag{68}$$

In this way, the propagator becomes, in zeroth order,

$$\langle q,m|\hat{P}\hat{U}^{\nu}|q',m'\rangle\approx\left[1-\frac{\pi^2}{6}m^2\right]^{\nu}\approx 1-\frac{\pi^2}{6}m^2\nu, \tag{69}$$

with $q'=0$; $m'=m$. Note that the main part of the propagator is diagonal. Then the equation of motion has the solution

$$\tilde{n}(m;\nu)=\left(1-\frac{\pi^2}{6}m^2\nu\right)\tilde{n}(m;0). \quad (70)$$

Taking the derivative of $\tilde{n}(m;\nu)$ twice with respect to m we get the MSD

$$\Sigma_\theta^2=-\frac{1}{4\pi^2}\frac{\partial^2\tilde{n}}{\partial m^2}\bigg|_{m=0}=\frac{1}{12}\nu. \quad (71)$$

(We have omitted a constant term, corresponding to the initial distribution).

Next we consider the first nontrivial choice of coefficients. Let

$$l_i=0,\quad k_1^1=1,\quad k_1^2=-1,\quad k_1^{i\neq 1,2}=0. \quad (72)$$

Note that for this choice $q_1=1$; $q_{i\neq 1}=0$. In the corresponding contributions to the sum (63) (we designate it by S_1) all but three Bessel functions contribute (in lowest order) with the factor 1. The three principal factors are that containing k_1^1, k_1^2, and q_1. From here it follows

$$S_1\approx J_0(K)J_1(2m)J_{-1}(2m)\approx -J_0(K)m^2. \quad (73)$$

The further procedure is analogous to that in the previous Sec. IV. For all S_i we get the same result, i.e., S_1. Summing all the terms S_i (remember that all of them belong to the same matrix element with $q'=0,\ m'=m$), we get

$$\sum_{i=1}^{\nu-2}(S_i+S_{-i})\approx 2\nu S_1\approx -2J_0(K)m^2\nu. \quad (74)$$

This is the first order term in the expansion of the propagator. Combining it with the zeroth order term, we can write a more precise equation of motion,

$$\tilde{n}(m;\nu)\approx\left(1-\frac{\pi^2}{6}m^2\nu-2J_0(K)m^2\nu\right)\tilde{n}(m;0), \quad (75)$$

from which (as $\nu\to\infty$)

$$\Sigma_\theta^2=\left(\frac{1}{12}+\frac{1}{\pi^2}J_0(K)\right)\nu, \quad (76)$$

$$D_\theta=2\pi^2\frac{\partial\Sigma_\theta^2}{\partial\nu}=\frac{\pi^2}{6}+2J_0(K) \quad (77)$$

follows. Expanding $J_0(K)$ for large arguments leads to

$$D_\theta=\frac{\pi^2}{6}+\sqrt{\frac{8}{\pi K}}\cos\left(K-\frac{\pi}{4}\right). \quad (78)$$

In Fig. 2 we compare this analytical result with numerical simulations. The agreement is excellent.

We conclude this section by some heuristic argument, why for the (in p) modulated map diffusion in θ should occur. We base the consideration on the assumption that for large K any p_i can be treated as an arbitrary function of θ_{i-1}. In other words, we assume the existence of a stationary action-density profile $[n_{st}(p)=\langle f(p,\theta)\rangle_\theta,(d/dt)n_{st}=0]$. p_i is just a random number, distributed in the interval $[-0.5,0.5]$. Let n_{st} have the first and second moments $\langle p\rangle=0$ and σ^2, respectively. Provided p is distributed uniformly, then the second moment is $\sigma^2=\frac{1}{12}$.

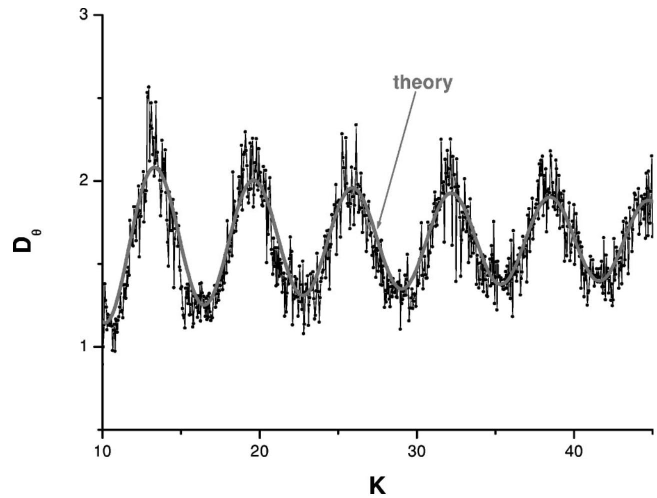

FIG. 2. Diffusion coefficient D_θ vs control parameter K for case B $(-0.5\leqslant p<+0.5)$. The dots (connected by lines) represent the measurements from numerical simulations while the solid curve shows the theoretical prediction.

A displacement of a given orbit at time ν is

$$\Delta\theta_\nu=\theta_\nu-\theta_0=\sum_{i=1}^{\nu}p_i\approx\theta_\nu. \quad (79)$$

According to the central limit theorem θ_ν should be distributed as

$$P(\theta_\nu)=\frac{1}{\tilde{\sigma}_\nu\sqrt{2\pi}}\exp\left(-\frac{\theta_\nu^2}{2\tilde{\sigma}_\nu^2}\right), \quad (80)$$

where $\tilde{\sigma}_\nu=\sigma\sqrt{\nu}$. From this distribution function the MSD can be easily obtained as

$$\Sigma_\theta^2=\int_{-\infty}^{\infty}\theta_\nu^2P(\theta_\nu)d\theta_\nu=\sigma^2\nu. \quad (81)$$

For a uniform p distribution (approached in the limit $K\to\infty$) the MSD becomes

$$\Sigma_\theta^2=\frac{1}{12}\nu. \quad (82)$$

So, we have a pure diffusive process in agreement with the previous formula. In general, the diffusion rate depends on the p distribution.

Comparing Eq. (81) with Eq. (76) we conclude that the second moment of the stationary distribution $n_{st}(p)$, i.e., the MSD of action, is an in fact slightly oscillating function of K: $\sigma^2=\frac{1}{12}+(1/\pi^2)J_0(K)$.

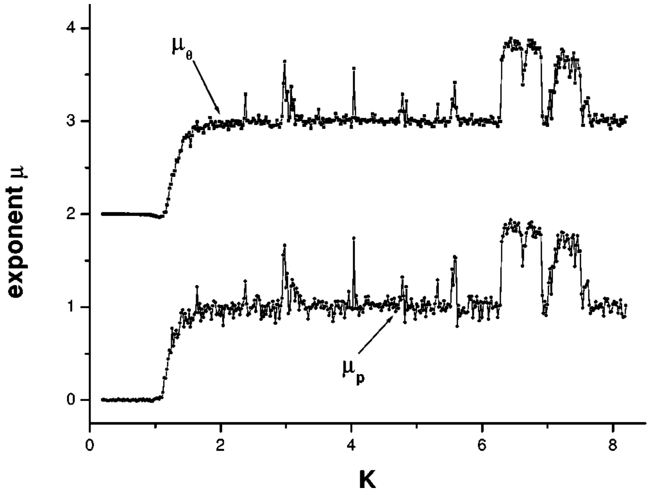

FIG. 3. Exponents μ_θ and μ_p vs K for case A $(-\infty<p<+\infty)$. Note that $\Sigma_\theta \sim \nu^{\mu_\theta}$ and $\Sigma_p \sim \nu^{\mu_p}$. The dots (connected by lines) represent the measurements from numerical simulations.

VI. SUMMARY AND DISCUSSION

In this paper we have investigated the angular transport in a nonperiodic Chirikov-Taylor Map. The results for the action transport, i.e., diffusive behavior in the limit of large stochasticity parameters K and transport barriers for $K<K_c$, are well-known. In the case of angular transport we have superdiffusive behavior for $K\to 0$. The transport properties for large K $(K\to\infty)$ depend on the boundary conditions on the action variable p. For an unrestricted p region (case A), the transport is superdiffusive with a transport exponent $\mu_\theta=3$ and $D_\theta \simeq \nu^2 D_p$. This theoretically predicted behavior is in complete agreement with numerical simulations, as shown in Fig. 3. On the other hand, for a periodic p behavior, the θ transport becomes diffusive, and the diffusion coefficient has been derived. Again, as shown in Fig. 4, the agreement with numerical simulations is excellent. Thus all analytical predictions are confirmed by numerical simulations. Figures 3 and 4 also contain the results for p diffusion.

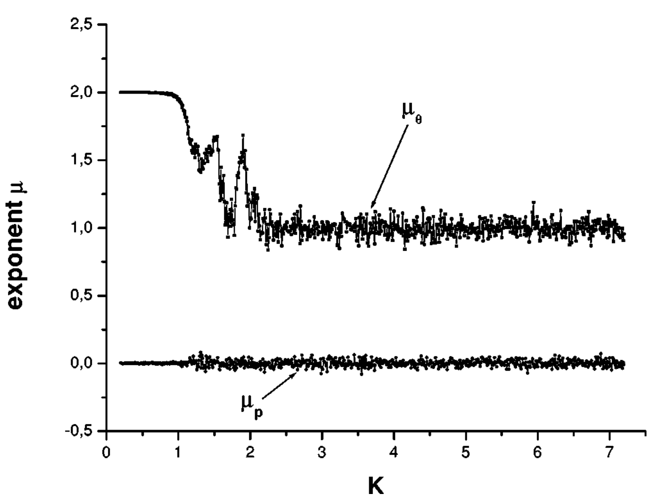

FIG. 4. Same as Fig. 3 but for case B $(-0.5\leq p<+0.5)$. In the latter case, obviously $\mu_p=0$.

Figure 3 shows that in case A the accelerator modes have the same influences on both, θ and p transport. On the other hand, in case B no divergences due to accelerator modes occur.

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APPENDIX A: SUPERDIFFUSION FOR $K=0$

Let us consider the continuous Hamiltonian system for the ‘‘kicked rotator’’

$$H(p,\theta;t)=\frac{p^2}{2}+\frac{K}{4\pi^2}\cos 2\pi\theta\sum_\nu \delta(t-\nu). \tag{A1}$$

The coordinates $p(t)$ and $\theta(t)$ coincide at $T=\nu-0$ with p_ν and θ_ν used in the main text. Here, ν stands for the iterated times in the standard map.

In the absence of perturbation $(K=0)$ the continuous system is trivial to integrate. The kinetic equation for the distribution function becomes

$$\frac{\partial f}{\partial t}+p\frac{\partial f}{\partial \theta}=0. \tag{A2}$$

In Fourier space, the corresponding equation

$$\frac{\partial \tilde{f}}{\partial t}-m\frac{\partial \tilde{f}}{\partial q}=0 \tag{A3}$$

has the solution

$$\tilde{f}(q,m;t)=\tilde{F}(m,mt+q), \tag{A4}$$

where $\tilde{F}$ is an arbitrary function that can be expressed through the initial distribution,

$$\tilde{f}(q,m;0):=\tilde{f}_0(q,m)=\tilde{F}(m,q). \tag{A5}$$

Thus,

$$\tilde{f}(q,m;t)=\tilde{f}_0(q+mt,m), \tag{A6}$$

and for the density profile we obtain

$$\tilde{n}(m;t)=\tilde{f}(q=0,m;t)=\tilde{f}_0(mt,m). \tag{A7}$$

Now Eq. (15) can be applied to estimate Σ_θ^2. Leaving only the dominant term in the large-time limit, it becomes

$$\Sigma_\theta^2 \to -\frac{1}{4\pi^2}\frac{\partial^2 \tilde{f}_0(q',m')}{\partial q'^2}\bigg|_{\substack{q'=0\\ m'=0}} t^2=\text{const }\ t^2. \tag{A8}$$

The first derivative of Σ_θ^2 with respect to time gives the running diffusion coefficient (16), i.e.,

$$D_\theta = -\frac{\partial^2 \tilde{f}_0(q',m')}{\partial q'^2}\bigg|_{\substack{q'=0\\m'=0}} t = \text{const}\ \ t. \tag{A9}$$

The regime is superdiffusive with the diffusion exponent $\mu_\theta = 2$.

APPENDIX B: RELATION BETWEEN ANGLE AND ACTION DIFFUSION

Let us start with the assumption that the transport in p direction is diffusive. Then we introduce $n(p;\nu)$ as the density profile at ''time'' ν. We designate $s(p'-p)$ as the transition probability, i.e., the probability that a ''orbit'' with momentum p after one iteration will have momentum p'. In an ideal diffusive process, $s(p)$ would be a Gaussian. Actually, not the shape but the width of $s(p)$ affects the dynamics. Although it is not exactly the case for the standard map, for the sake of simplicity we use

$$s(p) = \frac{1}{\sqrt{2\pi}\sigma} e^{-p^2/2\sigma^2}, \quad \tilde{s}(q) = e^{-2\pi^2\sigma^2 q^2}. \tag{B1}$$

Here σ^2 is an effective one-step mean square displacement of p. From the standard map one can see that $\sigma = \sqrt{\langle \Delta p\rangle^2} = \sqrt{\langle (K^2/4\pi^2)\sin(\theta)^2\rangle} \sim K/\pi\sqrt{8}$. The stickiness property leads to small (periodical) deviations from the linear dependence $\sigma(K) \sim K$. Having a density profile $n(p;\nu)$ at ''time'' ν, after one iteration it becomes

$$n(p;\nu+1) = \int dp' n(p';\nu) s(p-p'), \tag{B2}$$

or in Fourier space

$$\tilde{n}(q;\nu+1) = \tilde{n}(q;\nu)\tilde{s}(q). \tag{B3}$$

Starting with an initial distribution $n(p;0) = \delta(p)$, we get after ν iterations

$$\tilde{n}(q;\nu) = [\tilde{s}(q)]^\nu = e^{-2\pi^2\nu\sigma^2 q^2}. \tag{B4}$$

Thus for action diffusion, using Eq. (15), the MSD is

$$\Sigma_p^2 = \nu\sigma^2, \tag{B5}$$

as was expected for diffusive regime. From here the action diffusion coefficient

$$D_p = 2\pi^2\sigma^2 \tag{B6}$$

follows.

We now interpret this formula in the following way. Knowing the diffusion coefficient D_p, one can estimate the effective p displacement σ. The relation between the transports in p and in θ will be estimated from the second equation of the standard map,

$$\Delta\theta = \theta' - \theta = p'. \tag{B7}$$

When an ''orbit'' starts with $\theta_0 = 0$, then $n(\theta_1;1) = s(\theta_1)$ is nothing else but the probability for this ''orbit'' to have the coordinate θ_1 at time $\nu = 1$. If during ν steps the ''orbit'' had successive momenta $p_1, p_2, \ldots, p_\nu$, then its actual coordinate is $\theta_\nu = p_1 + p_2 + \cdots + p_\nu$. The amplitude of this process is $s(p_1)s(p_2-p_1)\cdots s(p_\nu - p_{\nu-1})$. Integrating over all intermediate states, we get the probability for the orbit to have the coordinate θ after ν iterations,

$$P_\nu(\theta) = \int \cdots \int dp_1 \cdots dp_{\nu-1} s(p_1) \times s(p_2-p_1)\cdots s(p_\nu - p_{\nu-1}), \tag{B8}$$

where

$$p_1 + p_2 + \cdots + p_\nu = \theta \tag{B9}$$

must hold.

After Fourier transform

$$\tilde{P}_\nu(q) = \tilde{s}(q)\tilde{s}(2q)\cdots\tilde{s}(\nu q). \tag{B10}$$

Substituting $\tilde{s}(q)$ from Eq. (B1) and using the summation formula

$$\sum_{i=1}^{\nu} i^2 = \frac{\nu^3}{3} + \frac{\nu^2}{2} + \frac{\nu}{6} \to \frac{\nu^3}{3}, \quad \text{as} \quad \nu \to \infty, \tag{B11}$$

we finally get

$$\tilde{P}_\nu(q) = \exp\left(-2\pi^2 \frac{\nu^3}{3}\sigma^2 q^2\right) = \exp(-2\pi^2\sigma_\nu^2 q^2), \tag{B12}$$

where $\sigma_\nu^2 = (\nu^3/3)\sigma^2$. Here $\tilde{P}_\nu(q)$ can be interpreted as a Fourier transform of the angular density profile. Using Eqs. (15) and (16), the MSD and the diffusion coefficient at ''time'' ν are

$$\Sigma_\theta^2 = -\frac{1}{4\pi^2}\frac{\partial^2}{\partial q^2}\tilde{P}_\nu(q)\bigg|_{q=0} = \frac{\nu^3}{3}\sigma^2, \tag{B13}$$

$$D_\theta = 2\pi^2\nu^2\sigma^2, \tag{B14}$$

respectively. Comparing Eq. (B6) with Eq. (B14), we conclude that

$$D_\theta = \nu^2 D_p. \tag{B15}$$

The relation $\mu_\theta = \mu_p + 2$ for the transport exponents was already found by Benkadda *et al.* [12].

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